

pected greater things from his pregnant and sagacious Wit. For he was scarce 20 Years of Age when he held these Correspondencies with Mr. *Crabtree*. And at the Age of 23. he was killed at *Marston-Moor-Battle*, on July 2. 1644. fighting for King *Charles I.* His Father was *Henry Gascoigne Esq;* of *Middleton*, between *Leeds* and *Wakefield*.

IV. *An Attempt towards the Improvement of the Method of approximating, in the Extraction of the Roots of Equations in Numbers.* By *Brook Taylor*, Secretary to the *Royal Society*.

I N *Phil. Tran.* No. 210. *Dr. Halley*, now Secretary of the *Royal Society*. has publish'd a very compendious and useful Method of extracting the Roots of affected Equations of the common Form, in Numbers. This Method proceeds by assuming the Root desired nearly true to one or two Places in Decimals (which is done by a Geometrical Construction, or by some other convenient way) and correcting the Assumption by comparing the Difference between the true Root and the assumed, by means of a new Equation whose Root is that Difference, and which he shews how to form from the Equation proposed, by Substitution of the Value of the Root sought, partly in known and partly in unknown Terms.

In doing this he makes use of a Table of Products (which he calls *Speculum Analyticum*,) by which he computes the Coefficients in the new Equation for finding the Difference mentioned. This Table, I observed, was formed in the same Manner from the Equation
pro-

propos'd, as the Fluxions are, taking the Root sought for the only flowing Quantity, its Fluxion for Unity, and after every Operation dividing the Product successively by the Numbers 1, 2, 3, 4, &c. Hence I soon found that this Method might easily and naturally be drawn from *Cor. 2. Prop. 7.* of my *Methodus Incrementorum*, and that it was capable of a further degree of Generality; it being Applicable, not only to Equations of the common Form, (*viz.* such as consist of Terms wherein the Powers of the Root sought are positive and integral, without any Radical Sign) but also to all Expressions in general, wherein any thing is proposed as given which by any known Method might be computed; if *vice versâ*, the Root were consider'd as given: such as are all Radical Expressions of Binomials, Trinomials, or of any other Nomial, which may be computed by the Root given, at least by Logarithms, whatever be the Index of the Power of that Nomial; as likewise Expressions of Logarithms, of Arches by the Sines or Tangents, of Areas of Curves by the *Abscissas* or any other Fluents, or Roots of Fluxional Equations, &c.

For the sake of this great Generality, it may not be improper to shew how this Method is derived from the foresaid *Corollary*. Therefore z and x being two flowing Quantities (whose Relation to one another may be expressed by any Equation whatsoever) by this *Corollary*, while z by flowing uniformly becomes $z + v$, x will

$$\text{become } x + \frac{\dot{x}}{1 \cdot z} v + \frac{\ddot{x}}{1 \cdot 2 z^2} v^2 + \frac{\ddot{\ddot{x}}}{1 \cdot 2 \cdot 3 z^3} v^3 + \&c.$$

$$\text{or } x + \frac{\dot{x} v}{1} + \frac{\ddot{x} v^2}{1 \times 2} + \frac{\ddot{\ddot{x}} v^3}{1 \cdot 2 \cdot 3} + \&c. \text{ for } z \text{ putting } 1.$$

Hence if y be the Root of any Expression formed of y and known Quantities, and supposed equal to nothing, and

and z be a part of y , and x be formed of z and the known Quantities, in the same manner as the Expression made equal to nothing is formed of y ; and let y be equal to $z + v$: the difference v will be found by Extracting

the Root of this expression $x + \frac{\dot{x}v}{1} + \frac{\ddot{x}v^2}{1.2} + \frac{\ddot{\dot{x}}v^3}{1.2.3} + \text{&c.} = 0$. For in this Case z being become $z + v = y$, x , which is now become $x + \dot{x}v + \frac{\ddot{x}v^2}{2} + \text{&c.}$

must become equal to nothing.

The Root v in the Equation $x + \frac{\dot{x}v}{1} + \frac{\ddot{x}v^2}{1.2} + \frac{\ddot{\dot{x}}v^3}{1.2.3} + \text{&c.} = 0$, is to be found upon the Supposition of its being very small with respect to z , (as it must be, if z be taken tolerably exact) by which means the Terms $\frac{\ddot{\dot{x}}v^3}{1.2.3} + \frac{\ddot{\ddot{x}}v^4}{1.2.3.4} + \text{&c.}$ may be neglected, upon account of their smallness with respect to the other Terms, so as to leave the Equation $x + \frac{\dot{x}v}{1} + \frac{\ddot{x}v^2}{1.2} = 0$, for finding the first approximation of v .

By extracting the Root of this Equation, we have

$$v = \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} - \frac{2x}{\ddot{x}} - \frac{\dot{x}}{\ddot{x}}}. \text{ That is,}$$

$$\text{First, } \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} - \frac{2x}{\ddot{x}} - \frac{\dot{x}}{\ddot{x}}}, \text{ if } x + \dot{x}v + \frac{\ddot{x}v^2}{2} = 0.$$

$$\text{Sec. } \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} + \frac{2x}{\ddot{x}} - \frac{\dot{x}}{\ddot{x}}}, \text{ if } -x + \dot{x}v + \frac{\ddot{x}v^2}{2} = 0.$$

Thirdly

$$3. \frac{x}{x} - \sqrt{\frac{x^2}{x^2} - \frac{2x}{x}}, \text{ if } x - xv + \frac{xv^2}{2}, \&c. = 0.$$

$$4. \frac{x}{x} - \sqrt{\frac{x^2}{x^2} + \frac{2x}{x}}, \text{ if } -x - xv + \frac{xv^2}{2}, \&c. = 0.$$

This approximation gives v exact to twice as many places as there are true Figures in z , and therefore triples the number of true Figures in the Expression of γ by $z + v$, which may be taken for a new Value of z , for computing a second v , seeking other Values of $x, \dot{x}, \ddot{x}, \&c.$ Tho' when z is tolerably exact (which it may be esteem'd when it contains two or three or more true Figures in the Value of γ , according to the Number of Figures the Root is propos'd to be computed to,) the Calculation may be restor'd without so much trouble,

only by taking $\sqrt{\frac{x^2}{x^2} + \frac{2x}{x} - \frac{2x^2}{2 \cdot 3 \cdot x} v^3 - \frac{2x^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot x} v^4$

$\&c.$ instead of $\sqrt{\frac{x^2}{x^2} + \frac{2x}{x}}$ taking every time for v its Value last computed.

From the same Equation $x + xv + \frac{xv^2}{2} + \frac{xv^3}{1 \cdot 2 \cdot 3} + \&c. = 0$, may be gather'd also a rational Form, *viz.*

$$v = \frac{-x}{x - \frac{xx}{2x}} \text{ For neglecting the Terms } \frac{xv^3}{1 \cdot 2 \cdot 3}, \&c.$$

we have $v = \frac{-x}{x + \frac{x}{2}v}$ which is nearly $= \frac{-x}{x}$. There-

fore in the Divisor instead of v writing $\frac{-x}{x}$ we have

more exactly $v = \frac{-x}{x - \frac{x^2}{2x}}$, that is

1. $\frac{-x}{x - \frac{x^2}{2x}}$, when $x + \dot{x}v + \frac{\ddot{x}v^2}{2} \text{ } \mathcal{C}c. = 0$.

2. $\frac{x}{x + \frac{x^2}{2x}}$, when $-x + \dot{x}v + \frac{\ddot{x}v^2}{2} \text{ } \mathcal{C}c. = 0$.

3. $\frac{x}{x - \frac{x^2}{2x}}$, when $x - \dot{x}v + \frac{\ddot{x}v^2}{2} \text{ } \mathcal{C}c. = 0$.

4. $\frac{-x}{x + \frac{x^2}{2x}}$, when $-x - \dot{x}v + \frac{\ddot{x}v^2}{2} \text{ } \mathcal{C}c. = 0$.

This *Formula* will also triplicate the number of true Figures in z . And the Calculation may be repeated, af-

ter every Operation, taking for a Divisor $x \pm \frac{x^2}{2} v \pm$

$\frac{\ddot{x}v^2}{2} + \frac{\ddot{x}v^3}{2} + \mathcal{C}c.$ instead of $x + \frac{x^2}{2x}$.

Dr. Halley has fully explain'd the manner of using both these *Formula's* in *Equations* of the common Form; wherefore I shall be the shorter in explaining two or three Examples of another sort.

Ex. I. Let it be proposed to find the Root of this Equation $y^2 + 1|v^2 + y - 16 = 0$. In this Case, for y writing z , and for 0 writing x , we have $z^2 + 1|v^2 + z$

+ z - 16 = x. Whence by taking the Fluxions, we have $\dot{x} = 2\sqrt{z} \times z \times z^2 + 1|^{\sqrt{z}-1} + 1$, and $\ddot{x} = 2\sqrt{z} \times 8 - 4\sqrt{z} z^2 \times z^2 + 1|^{\sqrt{z}-2}$. For finding the first Figures of the Root y , for \sqrt{z} take $\frac{7}{5}$, and we

have the Equation $y^2 + 1|^{\frac{7}{5}} + y - 16 = 0$, which being expanded gives $y^6 + 3y^4 + 2y^2 + 32y - 255 = 0$.

By this Equation I find that for the first supposition we may take $z = 2$. Therefore in order to find v , let us now make $\sqrt{z} = \frac{7}{5}$, (which is nearer than before)

and we have $x = z^2 + 1|^{\frac{7}{5}} + z - 16 = 2^2 + 1|^{\frac{7}{5}} - 14 = 5^{\frac{7}{5}} - 14 = -4,48$; $\dot{x} = 10,66$; $\ddot{x} = 4,72$. Whence

by the second rational Form $v = \frac{4,48}{10,66 + \frac{4,72 \times 4,48}{2 \times 10,66}}$

$= 0,38$; which must be too big, because $\frac{7}{5} < \sqrt{z}$, and therefore will require a larger Value of y to exhaust the Equation, than where \sqrt{z} is exact. For the second supposition therefore, let us take $z = 2,3$, and make $\sqrt{z} = 1,4142136$, and by help of the Logarithms we shall have $z^2 + 1|^{\sqrt{z}} = 13,47294$, whence $x = -0,22706$; $\dot{x} = 14,93429$, and $\ddot{x} = 5,18419$. Hence by the 2d.

irrational Formula $v = \sqrt{\frac{14,93429^2}{5,18419^2} + \frac{0,45412}{5,18419} - \frac{14,93429}{5,18419}}$ = 0,01516, which gives $y = z + v =$

$2,31516$, which is true to six Places. If you desire it more exact than to the extent of the Tables of Logarithms, taking $z = 2,31516$ for the next supposition, the Calculation must be repeated by computing of $z^2 + 1|^{\sqrt{z}}$ to a sufficient number of Places; which must be done by the Binomial Series, or by making a Logarithm

rithm on purpose, true to as many places as are necessary,

Ex. II. For another Example, let it be required to find the Number whose Logarithm is 0, 29, supposing we had no other Table of Logarithms but Mr. Sharps of 200 Logarithms to a great many places. This amounts to the resolving this Equation $ly = 0, 29$, or $ly - 0, 29 = 0$.

Hence therefore we have $x = lx - 0, 29$, $\dot{x} = \frac{a}{z}$ (a

being the *Modulus* belonging to the Table we use, *viz.*

$$0, 4342944819, \text{ \&ccaron.}) \ddot{x} = \frac{-a}{z^2}, \ddot{\ddot{x}} = \frac{2a}{z^3}, \ddot{\ddot{\ddot{x}}} = \frac{-6a}{z^4}$$

\&ccaron. In this Case because \ddot{x} has a negative Sign, changing the Signs of all the Coefficients, the Canon for v will be found in the fourth Case, which in the irrational Form

$$\text{gives } v = \frac{\dot{x}}{\ddot{x}} - \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} + \frac{2x}{\ddot{x}} - \frac{2\ddot{x}}{2.3\ddot{x}}v^3 - \frac{2\ddot{\ddot{x}}}{2.3.4\ddot{x}}v^4}$$

$$\text{\&ccaron.} = z - \sqrt{z^2 + \frac{2lz - 0, 58}{a} \times z^2 + \frac{2v^3}{3z} - \frac{2v^4}{4z^2}}$$

$$+ \frac{2v^5}{5z^3} \text{\&ccaron.}$$

In this Case to avoid often dividing by z , it will be most convenient to compute $\frac{v}{z}$, which is got

$$\text{from this Equation } \frac{v}{z} = 1 - \sqrt{1 + \frac{2lz - 0, 58}{a} +$$

$$\frac{2v^3}{3z^3} - \frac{2v^4}{4z^4} + \frac{2v^5}{5z^5}, \text{\&ccaron.}$$

The nearest Logarithm, in the Tables proposed, to the proposed Logarithm 0, 29 is 0, 2900346114, its Number being 1, 95. Therefore for the first supposition taking $z = 1, 95$, we have x

$$(\text{=} lz - 0, 29 = 0, 2900346114 - 0, 29) = 0, 0000$$

(617)

$$0,0000346114, \text{ and } \frac{2lz - 0,58}{a} = \frac{0,0000692228}{0,4342944819} =$$

$$0,00015939139, \text{ and } 1 + \frac{2lz - 0,58}{a} = 1,00015939139$$

Whence for the first approximation we have

$$\frac{v}{z} = 1 - \sqrt{1,00015939139} = -0,00007969247,$$

$$\text{and } v = -0,00015540032, \text{ and } y = z + v = 1,94984459968.$$

Which is true to eleven places, and may easily be corrected by the Terms $\frac{2v^3}{3z}$ &c. which I leave to the Readers curiosity.

Being upon the Subject of Approximations; it may not be amiss to set down here two Approximations I have formerly hit upon. The one is a Series of Terms for expressing the Root of any Quadratick Equation: and the other is a particular Method of Approximating in the invention of Logarithms, which has no occasion for any of the Transcendental Methods, and is expeditious enough for making the Tables without much trouble.

A general Series for expressing the Root of any Quadratick Equation.

Any Quadratick Equation being reduc'd to this Form $xx - mqx + my = 0$, the Root x will be express'd by this Series of Terms.

$$x = \frac{y}{q} + A \times \frac{1}{\frac{mq^2}{y} - 2} + B \times \frac{1}{a^2 - 2} + C \times \frac{1}{b^2 - 2} + D \times \frac{1}{c^2 - 2} \text{ \&c. Which must be thus interpreted.}$$

1. The Capital Letters A, B, C, &c. stand for the whole Terms with their Signs, preceding those where-
in

in they are found, as $B = A \times \frac{1}{\frac{mq^2}{y} - 2}$.

2. The little Letters $a, b, c,$ &c. in the Divisors, are equal to the whole Divisors of the Fraction in the Terms immediately preceding; thus $b = a^2 - 2$.

For an Example of this, let it be required to find $\sqrt{2}$. Putting $\sqrt{2} = x + 1$, we have $x^2 + 2x - 1 = 0$, which being compared with the general *Formula*, gives $mq = -2$, and $my = -1$: therefore for m taking -1 , we have $q = 2$, and $y = 1$, which Values substituted in the Series give $x = \frac{1}{2} - \frac{1}{2 \times 6} + \frac{1}{2 \times 6 \times 34}$

$-\frac{1}{2 \times 6 \times 34 \times 1154} + \frac{1}{2 \times 6 \times 34 \times 1154 \times 1331714}$, &c. The Fractions here wrote down giving the Root true to twenty three Places.

A new Method of computing Logarithms.

This Method is founded upon these Considerations.

1. That the Sum of the Logarithms of any two Numbers is the Logarithm of the Product of those two Numbers Multiplied together.

2. That the Logarithm of Unite is nothing; and consequently that the nearer any Number is to Unite, the nearer will its Logarithm be to 0. 3^{dly}. That the Product by Multiplication of two Numbers, whereof one is bigger, and the other less than Unite, is nearer to Unite than that of the two Numbers which is on the same side of Unite with its self; for Example the two Numbers being $\frac{3}{2}$ and $\frac{4}{3}$, the Product $\frac{8}{3}$ is less than Unite, but nearer to it than $\frac{3}{2}$, which is also less than Unite. Upon these Considerations, I found the present Approximation;

proximation; which will be best explain'd by an Example. Let it therefore be propos'd to find the Relation of the Logarithms of 2 and of 10. In order to this, I take two Fractions $\frac{128}{100}$ and $\frac{8}{10}$, viz. $\frac{2^7}{10^2}$ and $\frac{2^3}{10^1}$

whose Numerators are Powers of 2, and their Denominators Powers of 10; one of them being bigger, and the other less than 1. Having set these down in Decimal Fractions in the first Column of the Table annex'd, against them in the second Column I set A and B for their Logarithms, expressing by an Equation the manner how they are Compounded of the Logarithms of 2 and 10, for which I write $l2$ and $l10$. Then Multiplying the two Numbers in the first Column together, I have a third Number 1,024, against which I write C for its Logarithm, expressing likewise by an Equation in what manner C is formed of the foregoing Logarithms A and B. And in the same manner the Calculation is continued; only observing this *Compendium*, that before I Multiply the two last Numbers already got in the Table, I consider what Power of one of them must be used to bring the Product the nearest to Unite that can be. This is found, after we have gone a little way in the Table, only by Dividing the Differences of the Numbers from Unite one by the other, and taking the Quotient with the nearest, for the Index of the Power wanted. Thus the two last Numbers in the Table being 0,8 and 1,024, their Differences from Unit are 0,200 and 0,024; therefore

$\frac{0,200}{0,024}$ gives 9 for the Index; wherefore Multiplying the

ninth Power of 1,024 by 0,8, I have the next Number 0,990352031429, whose Logarithm is $D = 9C - B$. In seeking the Index in this manner by Division of the Differences, the Quotient ought generally to be taken with

with the least: but in the present case it happens to be the most, because instead of the Difference between 0, 8 and 1, we ought strictly to have taken the difference between the reciprocal 1, 25 and 1, which would have given the Index 105; and that would be too big, because the Product by that means would have been bigger than 1, as 1,024 is. Whereas this Approximation requires that the Numbers in the first Column be alternately greater and less than 1, as may be seen in the Table.

When I have in this manner continued the Calculation, till I have got the Numbers small enough, I suppose the last Logarithm to be equal to nothing. Which gives me an Equation, from which having got away the Letters by means of the foregoing Equations, I have the relation of the Logarithms proposed. In this manner if I suppose $G = 0$, I have $2136 / 2 - 643 / 10 = 0$. Which gives the Logarithm of 2 true in seven Figures, and too big in the Eighth; which happens because the Number corresponding with G is bigger than Unite.

There is another Expedient which renders this Calculation still shorter. It is founded upon this Consideration, that when x is very small $1 + x^n$ is very nearly $1 + nx$. Hence if $1 + x$, and $1 - z$ are the two last Numbers already got in the first Column of the Table, and their Powers $1 + x^m$ and $1 - z^n$ are such as will make the Product $1 + x^m \times 1 - z^n$ very near to Unite, m and n may be found thus: $1 + x^m = 1 + mx$, and $1 - z^n = 1 - nz$, and consequently $1 + x^m \times 1 - z^n = 1 + mx - nz - mnzx$, or (neglecting $mnzx$) $1 + mx - nz$. Make this equal to 1, and we have $m:n::z:x::l1 - z : l1 + x$. Whence $x l1 - z + z l1 + x = 0$. To give an Example of the Application of this, let 1,024 and 0,990352 be the last Numbers in the Table, their Logarithms being C and D. Then we have

$r, 024 = 1 + x$, and $0, 990352 = 1 - z$, and consequently $x = 0, 024$, and $z = 0, 006648$. Whence the Ratio $\frac{z}{x}$ in the least Numbers is $\frac{201}{500}$. So that for finding the

Logarithms proposed we may have $500 D + 201 C = 48510 l 2 - 14603 l 10 = 0$, which gives $l 2 = 0, 3010307$, which is too big in the last Figure; but it is nearer the truth, than what is got from the Logarithm F supposed equal to nothing. So that by this means we have saved four Multiplications, which were necessary to find the Number $9989595 \&c.$ correspondent to F, and which must have been had if we would make the Logarithm true to the same Number of places without this *Compendium*.

1, 280000000000	A = 7 l 2 - 2 l 10	_____	l 2 > 0, 28
0, 800000000000	B = 3 l 2 - 1 l 10	_____	< 0, 33
1, 024000000000	C = B + A = 10 l 2 - 3 l 10	_____	> 0, 300
0, 990352031429	D = 9 l 2 + B = 93 l 2 - 28 l 10	_____	< 0, 30107
1, 004336277664	E = 2 D + C = 196 l 2 - 59 l 10	_____	> 0, 301020
0, 998959536107	F = 2 E + D = 485 l 2 - 146 l 10	_____	< 0, 3010309
1, 000162894165	G = 4 F + E = 2136 l 2 - 643 l 10	_____	> 0, 30102996
0, 999936281874	H = 6 G + F = 13301 l 2 - 4004 l 10	_____	< 0, 301029997
1, 000035441215	I = 2 H + G = 28738 l 2 - 8651 l 10	_____	> 0, 301029991
0, 999971720830	K = I + H = 42039 l 2 - 12655 l 10	_____	< 0, 301029989
1, 000007161046	L = K + I = 70777 l 2 - 21306 l 10	_____	> 0, 30102999562
0, 999993203514	M = 3 L + K = 254370 l 2 - 76573 l 10	_____	< 0, 30102999567
1, 000000364511	N = M + L = 325147 l 2 - 97879 l 10	_____	> 0, 3010299956635
0, 999999764687	O = 18 N + M = 6107016 l 2 - 1838395 l 10	_____	< 0, 3010299956640

Com. Ar. 235313
 0 = 364511 O + 235313 N = 2302585825187 l 2 - 693147400972 l 10
 > 0, 301029995663987

I have computed this Table so far, that the Reader may see in what manner this Method Approximates; this whole Work, as it appears, costing a little more than three Hours time.

V. *Proprietates quaedam simplices Sectionum Conicarum ex natura Focorum deductæ; cum Theoremate generali de Viribus Centripetis; quorum ope Lex Virium Centripetarum ad Focos Sectionum tendentium, Velocitates Corporum in illis revolventium, & Descriptio Orbium facillime determinantur. Per Abr. de Moivre. R. S. Soc.*

SIT *DE* Axis Transversus Ellipseos, *AO* Axis alter, & *C* centrum Sectionis. Sit *P* punctum quodvis in circumferentia ejus; *PQ* Tangens curvæ ad *P*, occurrens Axi Transverso ad *Q*; puncta *S, F* Foci; *CP, CK* semidiametri Conjugatæ; *PH* Semilatus rectum ad diametrum *PC*; *PG* normalis ad Tangentem, cui occurrat *HG*, perpendicularis ipsi *PCH*, in puncto *G*, ut fiat *PG* radius Curvaturæ Ellipseos in puncto *P*: sint etiam *ST, CR, FV* perpendiculares in Tangentem *PQ* demissæ: Jungatur *SO*, & demittatur in Axem normalis *PL*. His positis, Dico quod,

I. *Rectangulum sub distantis ab utroque Ellipseos Foco, sive $SP \times PF$ aequale est quadrato Semidiametri CK .*

Demonstratio.

$$PSq = PCq + CSq - 2CS \times CL \text{ per 13. II. Elem.}$$

$$PFq = PCq + CSq + 2CS \times CL \text{ per 12. II. Elem.}$$

$$\text{Unde } PSq + PFq = 2PCq + 2CSq.$$

$$\text{Jam } PS + PF = DE = 2CD; \text{ ac propterea}$$

$$PSq + PFq + 2PS \times PF = 4CDq. \quad \text{Quare}$$